THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 9
February 11, 2025 (Tuesday)

1 Recall

We say K is a convex set if $\forall x, y \in K$, then

$$\lambda x + (1-\lambda)y \in K, \qquad \forall \lambda \in (0,1)$$

Proposition 1. *Let K be a closed convex subset of $\mathbb{R}^n$, and $y \in \mathbb{R}^n$ be fixed, there exists a **unique** $x^* \in K$ such that*

$$\|x^* - y\| = \min_{x \in K} \|x - y\|.$$

We define $\operatorname{Proj}_K(y) := x^*$.

2 Separation Theorem

Theorem 2. *Let $X \subseteq \mathbb{R}^n$ be a non-empty closed convex set and $y \notin X$, there exist $w = x^* - y \neq 0$, where $x^* = \operatorname{Proj}_X(y)$ such that*

$$\inf_{x \in X} w^T x = w^T x^* > w^T y.$$

To replace w by $-w$, we also have

$$\sup_{x \in X} w^T x < w^T y.$$

Corollary 3. *Let X_1, X_2 be two closed convex sets such that $X_1 \cap X_2 = \emptyset$ and X_2 is bounded, then there exists $0 \neq w \in \mathbb{R}^n$ such that*

$$\sup_{x_1 \in X_1} w^T x_1 < \inf_{x_2 \in X_2} w^T x_2.$$

Sketch of Proof. Note that $X_1 - X_2$ is closed (to use X_2 is bounded to conclude) and convex, and $0 \notin X_1 - X_2$.
Then, we have

$$\begin{aligned}
\sup_{x \in X_1 - X_2} w^T x < w^T 0 = 0 &\iff \sup_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} w^T(x_1 - x_2) < 0 \\
&\iff \sup_{x_1 \in X_1} w^T x_1 + \sup_{x_2 \in X_2} w^T(-x_2) < 0 \\
&\iff \sup_{x_1 \in X_1} w^T x_1 < \inf_{x_2 \in X_2} w^T x_2
\end{aligned}$$

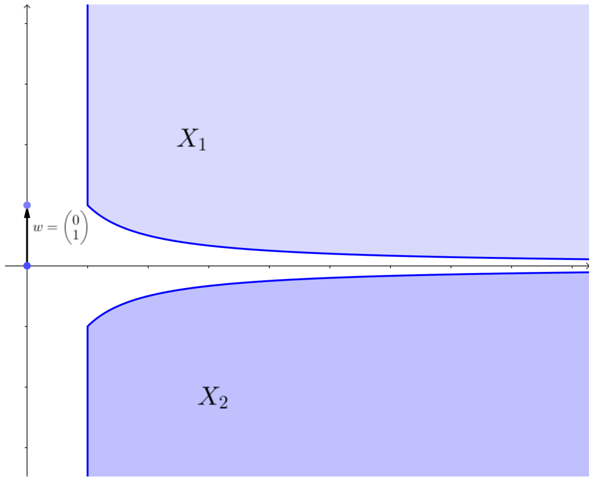
□

If X_2 is not bounded, then the inequality may not be strict. See the following counterexample. Consider

$$X_1 := \left\{(x,y) : x \geq 1,\ y \geq \frac{1}{x}\right\}$$

$$X_2 := \left\{(x,y) : x \geq 1,\ y \leq -\frac{1}{x}\right\}$$

 Prepared by Max Shung



In this case, we consider $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and then

$$\sup_{x \in X_1} \langle w, x \rangle = +\infty > -\infty = \inf_{x \in X_2} \langle w, x \rangle$$

and

$$\inf_{x_1 \in X_1} \langle w, x_1 \rangle = 0 = \sup_{x_2 \in X_2} \langle w, x_2 \rangle$$

Linear Form

Definition 1. We say a linear form $\langle w, x \rangle$ properly separate nonempty sets S and T if and only if

$$\sup_{x \in S} \langle w, x \rangle \leq \inf_{y \in T} \langle w, y \rangle \quad \text{and} \quad \inf_{x \in S} \langle w, x \rangle < \sup_{y \in T} \langle w, y \rangle$$

Definition 2. Let $X \subseteq \mathbb{R}^n$. $\mathrm{Lin}(X)$ be the affine subspace generated by X. The relative interior of X ($\mathrm{ri}(X)$) is the interior of X if it is considered as a subspace of $\mathrm{Lin}(X)$.

Theorem 4. *Let X_1, X_2 be two nonempty convex sets (not necessarily closed) such that*

$$\mathrm{ri}(X_1) \cap \mathrm{ri}(X_2) = \emptyset.$$

Then X_1 and X_2 can be properly separated.

Lemma 5. *Let $S \subseteq \mathbb{R}^n$ be a subset. Then*

$$\begin{aligned} S \text{ is separable} &\iff \text{there exists countable dense subset of } S \\ &\iff \text{there exist } K \subseteq S \text{ such that } K \text{ is countable and} \\ &\quad \forall x \in S, \text{ there exists } (x_n) \subseteq K,\ x_n \to x \end{aligned}$$

Proof. Let $(r_i)_{i=1,2,\dots}$ be the set of all rational vectors so that $(r_i)_{i=1,2\dots}$ is a countable dense subset of $\mathbb{R}^n$. Then, for each $n \geq 1$, we define a set K_n as follows:
For each r_i, we look for a point $x \in S$ such that $\|x - r_i\| \leq 1/n$.
If such $x \in S$ exist, we take one and add it in K_n so that K_n is countable.
Let $K = \bigcup_{n \geq 1} K_n \subseteq S$ and K is countable. Let $x \in S$ be an arbitrary point, there exists a sequence $r_n \to x$ where each r_n is a rational vector, then for any n, there exists $m \geq n$ such that one has $x_{m_n} \in K_{m_n} \subseteq K$ and $\|x_{m_n} - r_{m_n}\| \leq 1/m_n$.
This leads to $x_{m_n} \to x$ as $n \to +\infty$. so that K is a dense subset of S. □

— End of Lecture 9 —